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# Asymptotic Approximations for $n!$

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**Abstract**

Several approximations for $n!$ have recently appeared in the literature. We show here how these approximations can be derived by expansion of certain polynomials in inverse powers of $n$ and comparison with Stirling's asymptotic series. Some new approximations are also given. The same procedure is applied to generate approximations for the ratio of two gamma functions.

**Keywords:** Factorial function, asymptotic expansion, approximations

## 1. Introduction

The classical Stirling formula for the factorial function $n!$

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \qquad (n \to \infty)$$

is a useful approximation for large positive values of $n$ (not necessarily an integer). A slight improvement on this result is that due to Burnside [2]

$$n! \simeq \sqrt{2\pi}\, e^{-n-\frac{1}{2}} (n + \tfrac{1}{2})^{n+\frac{1}{2}}.$$

Recently, Mortici [3] has given the approximation

$$n! \simeq \sqrt{2\pi}\, e^{-n-\frac{1}{2}} (n^2 + n + \tfrac{1}{6})^{\frac{1}{2}n+\frac{1}{4}},$$

which he obtained by forming the geometric mean of the upper and lower bounds in a double inequality for $n!$. Numerical considerations show that when $n = 50$, for example, the relative error in this approximation is about 5 orders of magnitude better than in Stirling's formula. Approximations for $n!$ of a slightly different type have been considered by Batir [1] in the form

$$n! \simeq \sqrt{2\pi}\, n^n e^{-n} \left(n + \frac{1}{6} + \frac{1}{72n} - \frac{31}{6480n^2} - \frac{139}{155520n^3} + \frac{9871}{6531840n^4}\right)^{\frac{1}{2}} \quad (1.1)$$

and

$$n! \simeq \sqrt{2\pi}\, n^n e^{-n} \left(n^2 + \frac{n}{3} + \frac{1}{18} - \frac{2}{405n} - \frac{31}{9720n^2}\right)^{\frac{1}{4}}. \tag{1.2}$$

We show how all these approximations result from use of finite truncations of the well-known asymptotic expansion of $n!$ given by

$$n! \sim \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n} \sum_{k=0}^{\infty} (-)^k \gamma_k n^{-k} \qquad (n \to \infty), \tag{1.3}$$

where the first few Stirling coefficients $\gamma_k$ have the values

$$\gamma_0 = 1, \quad \gamma_1 = -\tfrac{1}{12}, \quad \gamma_2 = \tfrac{1}{288}, \quad \gamma_3 = \tfrac{139}{51840}, \quad \gamma_4 = -\tfrac{571}{2488320}, \quad \gamma_5 = -\tfrac{163879}{209018880}.$$

In addition, we demonstrate how similar approximations for the ratio of two gamma functions $\Gamma(n+1)/\Gamma(n+\frac{1}{2})$ for large positive $n$, also recently considered by Mortici [4], can be obtained by an analogous process. These ideas are extended to include an approximation for the ratio $\Gamma(n+\alpha)/\Gamma(n+\beta)$ for finite values of $\alpha$ and $\beta$.

## 2. Approximations

In [3], Mortici gave the double inequality satsified by $n!$

$$\sqrt{2\pi}\, e^{\sqrt{3}/6} \left(\frac{n+\omega_-}{e}\right)^{n+\frac{1}{2}} < n! < \sqrt{2\pi}\, e^{-\sqrt{3}/6} \left(\frac{n+\omega_+}{e}\right)^{n+\frac{1}{2}},$$

where $\omega_\pm = (3 \pm \sqrt{3})/6$. By forming the geometric mean between the upper and lower bounds, he considered the approximation

$$\begin{aligned} n! &\simeq \sqrt{2\pi}\, e^{-n-\frac{1}{2}} (n^2 + n + \tfrac{1}{6})^{\frac{1}{2}n+\frac{1}{4}} \\ &= \sqrt{2\pi}\, n^{n+\frac{1}{2}} e^{-n-\frac{1}{2}} \left(1 + \frac{1}{n} + \frac{1}{6n^2}\right)^{\frac{1}{2}n+\frac{1}{4}}. \end{aligned} \tag{2.1}$$

Since expansion of the series in inverse powers of $n$ yields

$$\begin{aligned} \left(1 + \frac{1}{n} + \frac{1}{6n^2}\right)^{\frac{1}{2}n+\frac{1}{4}} &= \exp\left[\left(\frac{n}{2} + \frac{1}{4}\right) \log\left(1 + \frac{1}{n} + \frac{1}{6n^2}\right)\right] \\ &= \exp\left[\frac{1}{2} + \frac{1}{12n} - \frac{1}{144n^3} + O(n^{-4})\right] \\ &= e^{\frac{1}{2}} \left(1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{71}{10368n^3} + O(n^{-4})\right), \end{aligned}$$

we see, by comparison with the asymptotic series for $n!$ in (1.3), that the approximation (2.1) is associated with an error of $O(n^{-3})$ for $n \to \infty$.

Similar approximations for $n!$ can be generated by considering

$$n! \simeq \sqrt{2\pi}\, e^{-n-\frac{1}{2}} H_r(n), \qquad H_r(n) = (P_r(n))^{m/r},$$

where we set $m \equiv n + \frac{1}{2}$, $P_r(n)$ denotes a polynomial of degree $r$ of the form

$$P_r(n) = n^r + a_1 n^{r-1} + \cdots + a_{r-1} n + a_r$$

and the coefficients $a_k$ $(1 \le k \le r)$ are to be determined. In order to remove the factor $e^{-1/2}$ we must set $a_1 = \frac{1}{2}r$. The coefficients $a_2, \ldots, a_r$ are then chosen so that the terms in the expansion of $H_r(n)$ in inverse powers of $n$ match the first $r$ terms of the asymptotic series in (1.3).

To illustrate the procedure we consider the case $r = 3$. We have

$$H_3(n) = (n^3 + \tfrac{3}{2}n^2 + a_2 n + a_3)^{m/3} = n^{n+\frac{1}{2}} \exp\left[\frac{m}{3}\log\left(1 + \frac{3}{2n} + \frac{a_2}{n^2} + \frac{a_3}{n^3}\right)\right]$$

$$= n^{n+\frac{1}{2}} e^{\frac{1}{2}} \left\{1 + \left(\frac{a_2}{3} - \frac{1}{8}\right)\frac{1}{n} + \left(\frac{25}{128} - \frac{3a_2}{8} + \frac{a_2^2}{18} + \frac{a_3}{3}\right)\frac{1}{n^2} + O(n^{-3})\right\}. \tag{2.2}$$

Comparison with (1.3) then yields $a_2 = \frac{5}{8}$ and $a_3 = \frac{1}{16}$, so that

$$H_3(n) = (n^3 + \tfrac{3}{2}n^2 + \tfrac{5}{8}n + \tfrac{1}{16})^{m/3}.$$

This yields the approximation

$$n! \simeq \sqrt{2\pi}\, e^{-n-\frac{1}{2}} (m^3 - \tfrac{1}{8}m)^{m/3}. \tag{2.3}$$

The term of $O(n^{-3})$ in (2.2) with the above values of $a_2$ and $a_3$ is $-5/(648n^3)$, with the result that the approximation (2.3) is, like that in (2.1), associated with an error of $O(n^{-3})$ for $n \to \infty$.

This procedure can be repeated in a straightforward manner for the case $r = 4$ and we find, with the help of *Mathematica*,

$$H_4(n) = (n^4 + 2n^3 + \tfrac{4}{3}n^2 + \tfrac{1}{3}n + \tfrac{2}{45})^{m/4}.$$

This produces the approximation

$$n! \simeq \sqrt{2\pi}\, e^{-n-\frac{1}{2}} (m^4 - \tfrac{1}{6}m^2 + \tfrac{17}{720})^{m/4}. \tag{2.4}$$

Since, for large $n$,

$$H_4(n) = n^{n+\frac{1}{2}} e^{\frac{1}{2}} \left\{1 + \frac{1}{12n} + \frac{1}{288n^2} - \frac{139}{51840n^3} - \frac{571}{2488320n^4} + O(n^{-5})\right\},$$

we see that the approximation (2.4) is associated with an error of $O(n^{-5})$. It is worth remarking that when $r$ is odd, the use of $r$ coefficients in $H_r(n)$ enables us to reproduce the first $r$ terms in the asymptotic series (1.3), whereas when $r$ is even we reproduce $r+1$ terms.

The introduction of additional, lower-order terms in $P_r(n)$ would lead to more accurate approximations, since this would enable more terms of the asymptotic series to be captured. For example, if we introduce the additional term $29/(1920n)$ in $P_3(n)$, we find

$$\begin{aligned} H_3(n) &= \left(n^3+\frac{3}{2}n^2+\frac{5}{8}n+\frac{1}{16}+\frac{29}{1920n}\right)^{m/3} \\ &= n^{n+\frac{1}{2}}e^{\frac{1}{2}}\left\{1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}+O(n^{-4})\right\}, \end{aligned}$$

which now results in an approximation that is associated with an error of $O(n^{-4})$, instead of $O(n^{-3})$, as $n\to\infty$. However, the inclusion of more lower-order terms, or increasing the index $r$, is ultimately illusory, since it then becomes just as easy to employ the asymptotic series (1.3).

The same procedure can be applied to generate approximations of the type in (1.1) and (1.2). If we consider the approximation

$$n! \simeq \sqrt{2\pi}\, n^n e^{-n}(P_r(n))^{1/(2r)}, \tag{2.5}$$

where $P_r(n)$ is the poynomial of degree $r$ defined above, we have

$$(P_r(n))^{1/(2r)} = n^{\frac{1}{2}}\left(1+\frac{a_1}{n}+\frac{a_2}{n^2}+\cdots+\frac{a_r}{n^r}\right)^{1/(2r)}.$$

Then, expansion of the term in brackets in inverse powers of $n$ and comparison with the asymptotic series (1.3) yields for $r=1,2,3,4$ the family of approximations given by

$$\begin{aligned} P_1(n) &= n+\tfrac{1}{6}, \\ P_2(n) &= n^2+\tfrac{1}{3}n+\tfrac{1}{18}, \\ P_3(n) &= n^3+\tfrac{1}{2}n^2+\tfrac{1}{8}n+\tfrac{1}{240}, \\ P_4(n) &= n^4+\tfrac{2}{3}n^3+\tfrac{2}{9}n^2+\tfrac{11}{405}n-\tfrac{8}{1215}. \end{aligned}$$

The associated error of the approximation (2.5) is found to be $O(n^{-r-1})$ as $n\to\infty$. For example, the case $r=4$ yields the approximation

$$n! \simeq \sqrt{2\pi}\, n^n e^{-n}(n^4+\tfrac{2}{3}n^3+\tfrac{2}{9}n^2+\tfrac{11}{405}n-\tfrac{8}{1215})^{\frac{1}{8}} \tag{2.6}$$

which has the expansion for $n\to\infty$

$$\sqrt{2\pi}\, n^{n+\frac{1}{2}}e^{-n}\left(1+\frac{1}{12n}+\frac{1}{288n^2}-\frac{139}{51840n^3}-\frac{571}{2488320n^4}+\epsilon\right),$$

where $\epsilon = 1561/(3317760n^5) + O(n^{-6})$, and so is accurate up to and including the term in $n^{-4}$. The first two polynomials are contained in (1.1) and (1.2), and $P_3(n)$ can be found in [7]. In (1.1) and (1.2), additional lower-order terms have been included in $P_r(n)$ to increase the accuracy. Straightforward expansion reveals that the error associated with (1.1) is $O(n^{-6})$ and that with (1.2) is $O(n^{-5})$ as $n \to \infty$.

In Table 1 we present values of the absolute relative error in the approximations (1.1), (2.1), (2.4) and (2.6) for different values of $n$.

| $n$ | Eq.(2.1) | Eq.(2.4) | Eq.(2.6) | Eq.(1.1) |
|---|---|---|---|---|
| 20 | $4.835 \times 10^{-7}$ | $3.772 \times 10^{-10}$ | $1.011 \times 10^{-10}$ | $1.682 \times 10^{-12}$ |
| 40 | $6.271 \times 10^{-8}$ | $1.254 \times 10^{-11}$ | $3.114 \times 10^{-12}$ | $2.996 \times 10^{-14}$ |
| 50 | $3.235 \times 10^{-8}$ | $4.161 \times 10^{-12}$ | $1.017 \times 10^{-12}$ | $8.048 \times 10^{-15}$ |
| 100 | $4.105 \times 10^{-9}$ | $1.333 \times 10^{-13}$ | $3.158 \times 10^{-14}$ | $1.318 \times 10^{-16}$ |
| 200 | $5.169 \times 10^{-10}$ | $4.218 \times 10^{-15}$ | $9.833 \times 10^{-16}$ | $2.107 \times 10^{-18}$ |

Table 1: The relative error associated with different approximations for $n!$.

## 3. Concluding remarks

The procedure of matching the coefficients of a polynomial of degree $r$ to an appropriate asymptotic series can also be applied to approximate the ratio of two gamma functions. Mortici [4] has given the approximations

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \simeq (P_r(n))^{1/(2r)}, \tag{3.1}$$

where the polynomials $P_r(n)$ of degree $r$ are[1]

$$\begin{aligned}
P_2(n) &= n^2 + \tfrac{1}{2}n + \tfrac{1}{8},\\
P_3(n) &= n^3 + \tfrac{3}{4}n^2 + \tfrac{9}{32}n + \tfrac{5}{128},\\
P_4(n) &= n^4 + n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{8}n,\\
P_5(n) &= n^5 + \tfrac{5}{4}n^4 + \tfrac{25}{32}n^3 + \tfrac{35}{128}n^2 + \tfrac{75}{2048}n + \tfrac{3}{8192},\\
P_6(n) &= n^6 + \tfrac{3}{2}n^5 + \tfrac{9}{8}n^4 + \tfrac{1}{2}n^3 + \tfrac{15}{128}n^2 + \tfrac{3}{256}n + \tfrac{11}{1024}.
\end{aligned}$$

These polynomial approximations can be obtained in the same manner as described in Section 2, by making use of the expansion [6, p. 51], [8, p. 68]

$$\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} \sim n^{\frac{1}{2}} \sum_{k=0}^{\infty} c_k n^{-k} \qquad (n \to \infty),$$

[1]The constant term in $P_4(n)$ is zero.

where the first few coefficients $c_k$ are

$$c_0 = 1, \quad c_1 = \tfrac{1}{8}, \quad c_2 = \tfrac{1}{128}, \quad c_3 = -\tfrac{5}{1024}, \quad c_4 = -\tfrac{21}{32768}, \quad c_5 = \tfrac{399}{262144}, \ldots\ .$$

For example, considering the case $r = 2$, we find

$$\begin{aligned}(P_2(n))^{\frac{1}{4}} &= (n^2 + a_1 n + a_2)^{\frac{1}{4}} \\ &= n^{\frac{1}{2}} \left(1 + \frac{a_1}{4n} + \frac{1}{4n^2}\left(a_2 - \frac{3a_1^2}{8}\right) + O(n^{-3})\right) \qquad (n \to \infty).\end{aligned}$$

Comparison with the above asymptotic series then shows that $a_1 = \frac{1}{2}$ and $a_2 = \frac{1}{8}$, whence we obtain the result stated. Routine expansion with the help of *Mathematica* reveals that the approximation (3.1) is associated with an error of $O(n^{-2\lfloor r/2\rfloor - 2})$ as $n \to \infty$, where $\lfloor\ \rfloor$ denotes the floor function. Thus, the approximation corresponding to $r = 4$ yields for large $n$

$$\begin{aligned}\frac{\Gamma(n+1)}{\Gamma(n+\frac{1}{2})} &\simeq (n^4 + n^3 + \tfrac{1}{2}n^2 + \tfrac{1}{8})^{\frac{1}{8}} \\ &= n^{\frac{1}{2}}\left(1 + \frac{1}{8n} + \frac{1}{128n^2} - \frac{5}{1024n^3} - \frac{21}{32768n^4} + \frac{399}{262144n^5} + \epsilon\right),\end{aligned}$$

where $\epsilon = -3227/(4194304n^6) + O(n^{-7})$, which is therefore associated with an error of $O(n^{-6})$.

The same argument can be used to show that, for finite values of $\alpha$ and $\beta$ (real or complex),

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} \simeq n^{\alpha-\beta-\frac{1}{2}}(n^2 + a_1 n + a_2)^{\frac{1}{4}}, \qquad a_1 = 4C_1,\ a_2 = 4C_2 + 6C_1^2,$$

which follows from the asymptotic expansion [5, p. 141], [6, p. 50]

$$\frac{\Gamma(n+\alpha)}{\Gamma(n+\beta)} \sim n^{\alpha-\beta}\left\{1 + \frac{C_1}{n} + \frac{C_2}{n^2} + \cdots\right\} \qquad (n \to \infty),$$

where

$$C_1 = \frac{1}{2}(\alpha-\beta)(\alpha+\beta-1), \quad C_2 = \frac{1}{24}(\alpha-\beta)(\alpha-\beta-1)[3(\alpha+\beta-1)^2 - (\alpha-\beta+1)].$$

When $\alpha = 1$, $\beta = \frac{1}{2}$, this approximation reduces to (3.1) with $r = 2$.

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